

A characterization of weighted local Hardy spaces

Lin Tang

Abstract In this paper, we give a characterization of weighted local Hardy spaces $h_\omega^1(\mathbb{R}^n)$ associated with local weights by using the truncated Reisz transforms, which generalizes the corresponding result of Bui in [1].

1. Introduction

The theory of local Hardy space plays an important role in various fields of analysis and partial differential equations; see [6, 1]. Bui [1] studied the weighted version h_ω^p of the local Hardy space h^p considered by Goldberg [6], where the weight ω is assumed to satisfy the condition (A_∞) of Muckenhoupt. R. Vyacheslav [11] introduced and studied some properties of the weighted local Hardy space h_ω^p spaces with weights that are locally in A_p but may grow or decrease exponentially. Recently, the author [10] established the weighted atomic decomposition characterizations of weighted local Hardy space h_ω^p with local weights.

The main purpose of this paper is to give a characterization of weighted local Hardy spaces $h_\omega^1(\mathbb{R}^n)$ associated with local weights by using the truncated Reisz transforms.

Throughout this paper, C denotes the constants that are independent of the main parameters involved but whose value may differ from line to line. Denote by \mathbb{N} the set $\{1, 2, \dots\}$ and by \mathbb{N}_0 the set $\mathbb{N} \cup \{0\}$. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$.

2. Statement of the main result

We first introduce weight classes A_p^{loc} from [11].

Let Q run through all cubes in \mathbb{R}^n (here and below only cubes with sides parallel to the coordinate axes are considered), and let $|Q|$ denote the volume of Q .

2000 Mathematics Subject Classification: 42B20, 42B25.
The research was supported by the NNSF (10971002) of China.

We define the weight class A_p^{loc} ($1 < p < \infty$) to consists of all nonnegative locally integral functions ω on \mathbb{R}^n for which

$$A_p^{loc}(\omega) = \sup_{|Q| \leq 1} \frac{1}{|Q|^p} \int_Q \omega(x) dx \left(\int_Q \omega^{-p'/p}(x) dx \right)^{p/p'} < \infty, \quad 1/p + 1/p' = 1. \quad (2.1)$$

The function ω is said to belong to the weight class of A_1^{loc} on \mathbb{R}^n for which

$$A_1^{loc}(\omega) = \sup_{|Q| \leq 1} \frac{1}{|Q|} \int_Q \omega(x) dx \left(\sup_{y \in Q} [\omega(y)]^{-1} \right) < \infty. \quad (2.2)$$

Remark: For any $C > 0$ we could have replaced $|Q| \leq 1$ by $|Q| \leq C$ in (2.1) and (2.2).

In what follows, $Q(x, t)$ denotes the cube centered at x and of the sidelength t . Similarly, given $Q = Q(x, t)$ and $\lambda > 0$, we will write λQ for the λ -dilate cube, which is the cube with the same center x and with sidelength λt . Given a Lebesgue measurable set E and a weight ω , let $\omega(E) = \int_E \omega dx$. For any $\omega \in A_\infty^{loc}$, L_ω^p with $p \in (0, \infty)$ denotes the set of all measurable functions f such that

$$\|f\|_{L_\omega^p} \equiv \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty,$$

and $L_\omega^\infty = L^\infty$. The space $L_\omega^{1,\infty}$ denotes the set of all measurable function f such that

$$\|f\|_{L_\omega^{1,\infty}} \equiv \sup_{\lambda > 0} \lambda \cdot \omega(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) < \infty.$$

We define the local Hardy-Littlewood maximal operator by

$$M^{loc} f(x) = \sup_{x \in Q: |Q| < 1} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Similar to the classical A_p Muckenhoupt weights, we give some properties for weights $\omega \in A_\infty^{loc} := \bigcup_{1 \leq p < \infty} A_p^{loc}$.

Lemma 2.1. *Let $1 \leq p < \infty$, $\omega \in A_p^{loc}$, and Q be a unit cube, i.e. $|Q| = 1$. Then there exists a $\bar{\omega} \in A_p$ so that $\bar{\omega} = \omega$ on Q and*

- (i) $A_p(\bar{\omega}) \leq C A_p^{loc}(\omega)$.
- (ii) if $\omega \in A_p^{loc}$, then there exists $\epsilon > 0$ such that $\omega \in A_{p-\epsilon}^{loc}(\omega)$ for $p > 1$.
- (iii) If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}^{loc} \subset A_{p_2}^{loc}$.
- (iv) $\omega \in A_p^{loc}$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}^{loc}$.
- (v) If $\omega \in A_p^{loc}$ for $1 \leq p < \infty$, then

$$\omega(tQ) \leq \exp(c_\omega t) \omega(Q) \quad (t \geq 1, |Q| = 1).$$

(vi) the local Hardy-Littlewood maximal operator M^{loc} is bounded on L_ω^p if $\omega \in A_p^{loc}$ with $p \in (1, \infty)$.

(vii) M^{loc} is bounded from L_ω^1 to $L_\omega^{1,\infty}$ if $\omega \in A_1^{loc}$.

We remark that Lemma is also true for $|Q| > 1$ with c depending now on the size of Q . In addition, it is easy to see that $A_p(\text{Munckenhout weight}) \subset A_p^{loc}$ for $p \geq 1$ and $e^{c|x|}$, $(1 + |x| \ln^\alpha(2 + |x|))^\beta \in A_1^{loc}$ with $\alpha \geq 0, \beta \in \mathbb{R}$ and $c \in \mathbb{R}$.

Let \mathcal{N} denote the class of C^∞ -functions φ on \mathbb{R}^n , supported on the cube $Q(0, 1)$ of center zero and half-side one whose mean value is not equal to zero. For $t > 0$, let $\varphi_t = t^{-2n} \varphi(z/t)$.

Given a distribution f , let $\varphi \in \mathcal{N}$, define the smooth maximal function by

$$\mathcal{M}f(z) = \sup_{0 < t < 1} |\varphi_t * f(z)|.$$

Follows from [10], we introduce the following weighted atoms.

Let $\omega \in A_1^{loc}$. A function a on \mathbb{R}^n is said to be a $(1, q)_\omega$ -atom for $1 < q \leq \infty$ if

- (i) $\text{supp } a \subset Q$,
- (ii) $\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(Q)]^{1/q-1}$.
- (iii) $\int_{\mathbb{R}^n} a(x) dx = 0$ if $|Q| < 1$.

Moreover, we call a is a $(1, q)_\omega$ single atom if $\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{1/q-1}$. we introduce weighted local Hardy spaces via smooth maximal functions and weighted local Hardy spaces. Moreover, we study some properties of these spaces.

The weighted local Hardy space is defined by

$$H_\omega^1(\mathbb{R}^n) \equiv \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : \mathcal{M}(f) \in L_\omega^1(\mathbb{R}^n) \right\}.$$

Moreover, we define $\|f\|_{h_\omega^1(\mathbb{R}^n)} \equiv \|\mathcal{M}(f)\|_{L_\omega^1(\mathbb{R}^n)}$. In [10], the author proved that

Theorem A. *Let $\omega \in A_1^{loc}$ and $1 < q \leq \infty$, then for any $f \in h_\omega^1(\mathbb{R}^n)$, there exists numbers λ_0 and $\{\lambda_i^k\}_{k \in \mathbb{Z}, i} \subset \mathbb{C}$, $(1, q)_\omega$ -atoms $\{a_i^k\}_{k \in \mathbb{Z}, i}$ with radius $r \leq 2$ and single atom a_0 such that*

$$f = \sum_{k \in \mathbb{Z}} \sum_i \lambda_i^k a_i^k + \lambda_0 a_0,$$

where the series converges almost everywhere and in $\mathcal{D}'(\mathbb{R}^n)$, moreover, there exists a positive constant C , independent of f , such that $\sum_{k \in \mathbb{Z}, i} |\lambda_i^k|^p + |\lambda_0|^p \leq C \|f\|_{h_\omega^1(\mathbb{R}^n)}^p$.

Let Φ be a non-negative, radial and C^∞ -function on \mathbb{R}^n with compact support $B(0, 2)$ and $\Phi \equiv 1$ on $B(0, 1)$. Define the truncated Riesz transforms by

$$R_j f(x) = \int_{\mathbb{R}^n} K_j(x - y) f(y) dy, \quad K_j(z) = \frac{z_j}{|z|^{n+1}} \Phi(z), \quad j = 1, \dots, n.$$

Now let us state the main result of this paper.

Theorem 1. *Let $\omega \in A_1^{loc}$. Then a function f is in $h_\omega^1(\mathbb{R}^n)$ if and only if $f \in L_\omega^1(\mathbb{R}^n)$ and $R_j f \in L_\omega^1(\mathbb{R}^n)$, $j = 1, \dots, n$. More precisely,*

$$\|f\|_{h_\omega^1(\mathbb{R}^n)} \sim \|f\|_{L_\omega^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j f\|_{L_\omega^1(\mathbb{R}^n)}.$$

We remark that if $\omega \in A_1$, then Theorem 1 has been proved in [1], that is,

Theorem B. *Let $\omega \in A_1$. Then a function f is in $h_\omega^1(\mathbb{R}^n)$ if and only if $f \in L_\omega^1(\mathbb{R}^n)$ and $R_j f \in L_\omega^1(\mathbb{R}^n)$, $j = 1, \dots, n$. More precisely,*

$$\|f\|_{h_\omega^1(\mathbb{R}^n)} \sim \|f\|_{L_\omega^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j f\|_{L_\omega^1(\mathbb{R}^n)}.$$

3. Proof of Theorem 1

Theorem 1 will be deduced by the following lemmas.

Lemma 3.1. *Let $\omega \in A_1^{loc}$. Then*

$$\|f\|_{h_\omega^1(\mathbb{R}^n)} \leq C(\|f\|_{L_\omega^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j f\|_{L_\omega^1(\mathbb{R}^n)}). \quad (3.1)$$

Proof. We will borrow some idea from [9]. Let Q is a unit cube, χ'_{3Q} is a C_0^∞ nonnegative function supported in $4Q$ and $\chi'_{3Q} = 1$ on $3Q$. By Lemma 2.1, we can set $\bar{\omega} \in A_p$ so that $\bar{\omega} = \omega$ on $14Q$. Fix $\varphi \in \mathcal{N}$, by Theorem B, we have

$$\begin{aligned} \left\| \sup_{0 < t < 1} |\varphi_t * f| \right\|_{L_\omega^1(Q)} &= \left\| \sup_{0 < t < 1} |\varphi_t * (f \chi'_{3Q})| \right\|_{L_{\bar{\omega}}^1(\mathbb{R}^n)} \\ &\leq C \|f \chi'_{3Q}\|_{h_{\bar{\omega}}^1(\mathbb{R}^n)} \\ &\leq C \left(\|f \chi'_{3Q}\|_{L_{\bar{\omega}}^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j(f \chi'_{3Q})\|_{L_{\bar{\omega}}^1(\mathbb{R}^n)} \right). \end{aligned} \quad (3.2)$$

On the other hand, by the properties of A_1^{loc} , we obtain

$$\begin{aligned} &\|R_j(f \chi'_{3Q}) - \chi'_{3Q} R_j(f)\|_{L_{\bar{\omega}}^1(\mathbb{R}^n)} \\ &\leq \left\| \int |R_j(z - y)[\chi'_{3Q}(y) - \chi'_{3Q}(z)]f(y)| \chi'_{12I}(y) dy \right\|_{L_{\bar{\omega}}^1(\mathbb{R}^n)} \\ &\leq C \int_{\mathbb{R}^n} \bar{\omega}(z) \int_{\mathbb{R}^n} |R_j(z - y)| |z - y| |f(y)| \chi'_{12I}(y) dy dz \\ &\leq C \|f\|_{L_{\bar{\omega}}^1(14Q)}. \end{aligned} \quad (3.3)$$

Combing (3.2) and (3.3), we obtain

$$\left\| \sup_{0 < t < 1} |\varphi_t * f| \right\|_{L^1_\omega(Q)} \leq C \left(\|f\|_{L^1_\omega(14Q)} + \sum_{j=1}^n \|R_j(f)\|_{L^1_\omega(6Q)} \right).$$

Summing on Q , we obtain (3.1).

Lemma 3.2. *Let R_j be as above, then*

$$(i) \quad \|R_j f\|_{L^p_\omega(\mathbb{R}^n)} \leq C_{p,\omega} \|f\|_{L^p_\omega(\mathbb{R}^n)} \text{ for } 1 < p < \infty \text{ and } \omega \in A_p^{loc}.$$

$$(ii) \quad \|R_j f\|_{L^{1,\infty}_\omega(\mathbb{R}^n)} \leq C_\omega \|f\|_{L^1_\omega(\mathbb{R}^n)} \text{ for } \omega \in A_1^{loc}.$$

Proof. We first note that for $\omega \in A_p$ the inequality (i) is known to be true, see [5]. For $\omega \in A_p^{loc}$, by Lemma 2.1 (i) for any unit cube Q there is a $\bar{\omega} \in A_p$ so that $\bar{\omega} = \omega$ on $6Q$. Then

$$\begin{aligned} \|R_j f\|_{L^p_\omega(Q)} &= \|R_j(\chi_{6Q} f)\|_{L^p_\omega(Q)} \\ &\leq \|R_j(\chi_{6Q} f)\|_{L^p_{\bar{\omega}}(Q)} \\ &\leq C \|(\chi_{6Q} f)\|_{L^p_{\bar{\omega}}(\mathbb{R}^n)} \\ &\leq C \|f\|_{L^p_{\bar{\omega}}(6Q)}. \end{aligned}$$

Summing over all dyadic unit Q gives (i).

For (ii), similar to (i), note that for $\omega \in A_1$ the inequality (ii) is known to be true, see [2]. Since $\omega \in A_1^{loc}$, by Lemma 2.1 (i) for any unit cube Q there is a $\bar{\omega} \in A_1$ so that $\bar{\omega} = \omega$ on $6Q$. Then for any $\lambda > 0$

$$\begin{aligned} \omega(\{x \in Q : |R_j f(x)| > \lambda\}) &\leq \omega(\{x \in Q : |R_j(\chi_{6Q} f)(x)| > \lambda\}) \\ &= \bar{\omega}(\{x \in Q : |R_j(\chi_{6Q} f)(x)| > \lambda\}) \\ &\leq C \lambda^{-1} \|(\chi_{6Q} f)\|_{L^1_{\bar{\omega}}(\mathbb{R}^n)} \\ &= C \lambda^{-1} \|f\|_{L^1_{\bar{\omega}}(6Q)}. \end{aligned}$$

Summing over all dyadic unit Q gives (ii).

Lemma 3.3. *Let $\omega \in A_1^{loc}$. Then*

$$\|R_j f\|_{h^1_\omega(\mathbb{R}^n)} \leq C \|f\|_{h^1_\omega(\mathbb{R}^n)}. \quad (3.4)$$

Proof: We first fix a function $\varphi \in \mathcal{N}$. Let $a(x)$ be a $(1, 2)$ atom in $h_\omega^1(\mathbb{R}^n)$, supported in a cube Q centered at y_0 and sidelength $r \leq 2$, or $a(x)$ is a $(1, 2)$ single atom. To prove the (iii), by Theorem A and Theorem 6.2 in [10], it is enough to show that

$$\|\mathcal{M}(R_j a)\|_{L_\omega^1(\mathbb{R}^n)} \leq C, \quad (3.5)$$

where C is independent of a .

If a is a single atom, by $L_\omega^2(\mathbb{R}^n)$ boundedness of \mathcal{M} and R_j , then

$$\|\mathcal{M}(R_j a)\|_{L_\omega^1(\mathbb{R}^n)} \leq C \|R_j a\|_{L_\omega^2(\mathbb{R}^n)} \omega(\mathbb{R}^n)^{1/2} \leq C.$$

Next we always assume that a is an atom in $h_\omega^1(\mathbb{R}^n)$, supported in a cube Q centered at y_0 and sidelength $r \leq 2$.

We first consider the atom a with sidelength $1 \leq r \leq 2$. Then by $L_\omega^2(\mathbb{R}^n)$ of the operators \mathcal{M} and R_j (see Lemma 3.2), we have

$$\begin{aligned} \|\mathcal{M}(R_j a)\|_{L_\omega^1(\mathbb{R}^n)} &= \int_{8Q} \mathcal{M}(R_j a)(x) \omega(y) dy \\ &\leq C \omega(8Q)^{1/2} \|a\|_{L_\omega^2(\mathbb{R}^n)} \leq C. \end{aligned}$$

If $r < 1$, we write

$$\begin{aligned} \|\mathcal{M}(R_j a)\|_{L_\omega^1(\mathbb{R}^n)} &= \int_{2Q} \mathcal{M}(R_j a)(x) \omega(y) dy + \int_{\mathbb{R}^n \setminus 2Q} \mathcal{M}(R_j a)(x) \omega(y) dy \\ &:= I + II. \end{aligned}$$

For I , by $L_\omega^2(\mathbb{R}^n)$ boundedness of the operators \mathcal{M} and R_j , we have

$$I \leq \omega(2Q)^{1/2} \|a\|_{L_\omega^2(\mathbb{R}^n)} \leq C.$$

We now estimate II . Let $x \notin 2Q$. For $t > 0$ we define the smooth functions

$$R_j^t = \varphi_t * K_j$$

and we observe that they satisfy

$$\sup_{0 < t < 1} |\partial^\beta K_j^t(x)| \leq C |x - y_0|^{-n-|\beta|} \chi_{\{|x-y_0| \leq 8n\}}(x) \quad (3.6)$$

for all $|\beta| \leq 1$; see their proof in page 507 of [7].

Now note that if $x \notin 2Q$ and $y \in Q$, then $|x - y_0| \geq 2|y - y_0|$ stays away from y_0 and $K_j(x - y)$ is well defined. We have

$$R_j a * \varphi_t(x) = (a * K_j^t)(x) = \int_Q K_j^t(x - y) a(y) dy.$$

Using the cancellation of atoms we deduce

$$\begin{aligned} R_j a * \varphi_t(x) &= \int_Q K_j^t(x-y)a(y)dy \\ &= \int_Q [K_j^t(x-y) - K_j^t(x-y_0)] a(y)dy \\ &= \int_Q \left[\sum_{|\beta|=1} (\partial^\beta K_j^t(x-y_0 - \theta_y(y-y_0))y^\beta \right] a(y)dy \end{aligned}$$

for some $0 \leq \theta_y \leq 1$. Using that $|x-y_0| \geq 2|y-y_0|$ and (3.6) we get

$$\begin{aligned} R_j a * \varphi_t(x) &\leq C|x-y_0|^{-n-1} \chi_{\{|x-y_0| \leq 8n\}}(x) \int_Q |a(y)||y|dy \\ &\leq C \frac{r^{n+1}}{|x-y_0|^{n+1}} \omega(Q)^{-1} \chi_{\{|x-y_0| \leq 8n\}}(x). \end{aligned} \tag{3.7}$$

By (3.7) and using properties of A_1^{loc} , we obtain

$$\begin{aligned} II &\leq C \int_{2r \leq |x-y_0| \leq 8n} \frac{r^{n+1}}{|x-y_0|^{n+1}} \omega(Q)^{-1} \omega(x) dx \\ &\leq C \frac{|Q|}{\omega(Q)} \sum_{k=1}^{k_0} 2^{-k} \frac{\omega(2^k Q)}{|2^k Q|} \leq C, \end{aligned}$$

where k_0 is an integer such that $8n \leq 2^{k_0} \leq 16n$.

Thus, (3.5) holds. Hence, the proof is complete.

Next, we study weighted $h_\omega^1(\mathbb{R}^n)$ boundedness for strongly singular integrals.

Given a real number $\theta > 0$ and a smooth radial cut-off function $v(x)$ supported in the ball $\{x \in \mathbb{R}^n : |x| \leq 2\}$, we consider the strongly singular kernel

$$k(x) = \frac{e^{i|x|^{-\theta}}}{|x|^n} v(x).$$

Let us denote by Tf the corresponding strongly singular integral operator:

$$Tf(x) = p.v \int_{\mathbb{R}^n} k(x-y)f(y)dy.$$

This operator has been studied by several authors, see [8], [12] and [4]. In particular, S. Chanillo [2] established the weighted $L_\omega^p(\mathbb{R}^n)$ ($\omega \in A_p, 1 < p < \infty$) and $H_\omega^1(\mathbb{R}^n)$ ($\omega \in A_1$) boundedness for strongly singular integrals. The author [10] proved the following results for the strongly singular integrals.

Theorem C. *Let T be strongly singular integral operators, then*

$$(i) \quad \|Tf\|_{L_\omega^p(\mathbb{R}^n)} \leq C_{p,\omega} \|f\|_{L_\omega^p(\mathbb{R}^n)} \text{ for } 1 < p < \infty \text{ and } \omega \in A_p^{loc}.$$

(ii) $\|Tf\|_{L^{1,\infty}_\omega(\mathbb{R}^n)} \leq C_\omega \|f\|_{L^1_\omega(\mathbb{R}^n)}$ for $\omega \in A_1^{loc}$.

(iii) $\|Tf\|_{L^1_\omega(\mathbb{R}^n)} \leq C_\omega \|f\|_{h^1_\omega(\mathbb{R}^n)}$ for $\omega \in A_1^{loc}$.

Theorem 1, Lemma 3.3 and (iii) in Theorem C imply immediately that

Corollary 1. *Let T be strongly singular integral operators, then*

$$\|Tf\|_{h^1_\omega(\mathbb{R}^n)} \leq C_\omega \|f\|_{h^1_\omega(\mathbb{R}^n)}$$

for $\omega \in A_1^{loc}$.

References

- [1] H. Bui, Weighted Hardy spaces, Math. Nachr. 103 (1981), 45–62.
- [2] S. Chanillo, Weighted norm inequalities for strongly singular convolution operators, Trans. Amer. Math. Soc. 281(1984), 77-107.
- [3] J. García-Cuerva, Weighted H^p spaces, Dissertationes Math. 162(1979), 63.
- [4] J. García-Cuerva, E. Harboure, S. Segovia and J. L. Torrea, Weighted norm inequalities for commutators of strongly singular integral, Indiana. Univ. Math. J. 40(1991), 1397-1420.
- [5] J. García-Cuerva and J. Rubio de Francia, Weighted norm inequalities and related topics, Amsterdam- New York, North-Holland, 1985.
- [6] D. Goldberg, A local version of real Hardy spaces, Duke Math. 46(1979), 27-42.
- [7] L. Grafakos, Classical and modern fourier analysis, 2004.
- [8] I. Hirschman, Multiplier transformations, Duke Math. J. 26(1959), 222-242.
- [9] G. Mauceri, M. Picardello and F. Ricci, A Hardy space associated with twisted convolution, Adv. Math. 39(1981), 270-288.
- [10] L. Tang, Weighted local Hardy spaces and their applications, preprint.
- [11] R. Vyacheslav, Littlewood-Paley theory and function spaces with A_p^{loc} weights, Math. Nachr. 224 (2001), 145–180.
- [12] S. Wainger, Special trigonometric series in k dimensions, Mem. Amer. Math. Soc. 59(1965).

LMAM, School of Mathematical Science
Peking University
Beijing, 100871
P. R. China

E-mail address: tanglin@math.pku.edu.cn